Math 255B Lecture 24 Notes

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March 4, 2020

1 The Spectral Theorem for Unbounded, Self-Adjoint Operators

1.1 Multiplicative properties of the functional calculus for bounded Baire functions

Last time, we set about extending our functional calculus to the class of bounded **Baire functions** $\mathrm{Ba}(\mathbb{R})$, the smallest class of functions $\mathbb{R} \to \mathbb{C}$ containing $C(\mathbb{R})$ which is closed under pointwise limits. We showed that for any $\varphi \in \mathrm{Ba}_b(\mathbb{R})$, there is a unique map $\varphi(A) \in \mathcal{L}(H, H)$ such that $\langle \varphi(A)u, v\rangle = \int \varphi(\lambda)\, d\mu_{u,v}(\lambda)$.

As in the case for continuous functions, we get

$$\|\varphi(A)\| \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}.$$

If $\varphi \in \mathrm{Ba}_b(\mathbb{R})$ is real, then $\langle \varphi(A)u, u\rangle \in \mathbb{R}$, so $\varphi(A)$ is self-adjoint. In general, $\varphi(A)^* = \overline{\varphi}(A)$.

Next, we have the multiplicative property:

Proposition 1.1. *Let A be self-adjoint, and let $\varphi, \psi \in \mathrm{Ba}_b(\mathbb{R})$. Then*

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

Proof. We may assume that φ, ψ are real. Fix $\varphi \in C_B$ and consider $K_M = \{\psi \in \mathrm{Ba}(\mathbb{R};\mathbb{R}) : |\psi| \leq M, \text{mult. prop. holds}\}$. Then K_M contains the continuous functions, and K_M is closed under pointwise convergence: if $\psi_j \in K_M$ with $\psi_j \to \psi$ pointwise, then $\psi_j(A) \to \psi(A) \in K_M$ weakly, so $(\varphi\psi_j)(A) \to (\varphi\psi)$ weakly. It follows that $K_M = \{\psi \in \mathrm{Ba}(\mathbb{R};\mathbb{R}) : |\psi| \leq M\}$. Next, keeping $\psi \in \mathrm{Ba}_b(\mathbb{R};\mathbb{R})$ fixed, we extend the multiplicative property to all $\varphi \in \mathrm{Ba}_b$. $\square$

It follows as in the continuous case that if $\varphi_j \in \mathrm{Ba}_b(\mathbb{R})$ with $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly (for all $u \in H$, $\|\varphi_j(A)u - \varphi(A)u\| \to 0$).

Remark 1.1. Assume that $\varphi \in \mathrm{Ba}_b$ is such that $\psi(\lambda) = \lambda\varphi(\lambda) \in \mathrm{Ba}_b$. Let

$$r(\lambda) = \frac{1}{z-\lambda},$$

$$\operatorname{Im} z \neq 0, \qquad \varphi_z(\lambda) = r(\lambda)^{-1}\varphi(\lambda) = \psi(\lambda) - z\varphi(\lambda) \in \mathrm{Ba}_b\,.$$

We can write

$$\varphi(\lambda) = r(\lambda)\varphi_z(\lambda),$$

so

$$\varphi(A) = r(A)\varphi_z(A).$$

Now $r(A) = R(z)$ ($\lambda r(\lambda) \in C_B$, so $Ar(A) = \frac{\lambda}{\lambda - z}(A) = 1 + zr(A)$). We get that $\operatorname{Im}\varphi(A) \subseteq D(A)$ and

$$A\varphi(A) = \underbrace{AR(z)}_{!+zR(z)}\varphi_z(A) = \psi(A).$$

So we get that the multiplicative property holds when one of the functions is λ (which is unbounded), as long as the result is still bounded.

1.2 The spectral theorem for unbounded, self-adjoint operators

Theorem 1.1 (spectral theorem)**.** *Given a self-adjoint operator A, there is a unique algebra homomorphism $\Phi : \mathrm{Ba}_b(\mathbb{R}) \to \mathcal{L}(H, H)$ sending $\varphi \mapsto \varphi(A)$ such that:*

1. $\overline{\varphi}(A) = \varphi(A)^*$
2. $\|\varphi(A)\| \leq \|\varphi\|_{L^\infty(\mathrm{Spec}(A))}$
3. *If $\varphi_j \in \mathrm{Ba}_b$ with $\varphi_j \to \varphi$ pointwise boundedly, then $\varphi_j(A) \to \varphi(A)$ strongly.*
4. *If $\varphi, \varphi_1(\lambda)$ $(= \lambda\varphi(\lambda)) \in \mathrm{Ba}_b$, then $\varphi_1(A) = A\varphi(A)$.*
5. *$u \in D(A)$ if and only if there is a uniform upper bound for $\|\varphi_1(A)u\|$ when $\varphi_1(\lambda) = \lambda\varphi(\lambda)$, $0 \leq \varphi \leq 1$, and $\varphi \in \mathrm{Ba}_b$ with compact support. In this case,*

$$Au = \lim_{j\to\infty} \varphi_{1,j}(A)u,$$

   where $\varphi_j \uparrow 1$.

Proof. Only uniqueness remains to be checked. Let $r_z(\lambda) = \frac{1}{\lambda - z}$, where $\operatorname{Im} z \neq 0$. Then $\lambda r_z(\lambda) = 1 + zr_z(\lambda)$, so by property 4, $A\Phi(r_z) = 1 + z\Phi(r_z)$. So we conclude that $\Phi(r_z) = R(z)$, the resolvent of A. Given $u \in H$, let μ be the measure on $\mathbb{R}$ sending $\varphi \mapsto \langle \Phi(\varphi)u, u\rangle$. Then

$$\langle \Phi(r_z)u, u\rangle = \langle R(z)u, u\rangle = \int \frac{1}{\lambda - z}\, d\mu(\lambda).$$

By the proof of Nevanlinna's theorem,

$$d\mu = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \operatorname{Im} \langle R(\lambda + i\varepsilon)u, u\rangle \, d\lambda,$$

which shows the uniqueness. □

1.3 Extending the functional calculus to unbounded Baire functions

Next, we will define $\varphi(A)$ as an unbounded operator when $\varphi \in \mathrm{Ba}(\mathbb{R})$. Recall that if $\varphi \in \mathrm{Ba}_b$, then

$$\|\varphi(A)u\|^2 = \int |\varphi(\lambda)|^2 \, d\mu_u(\lambda).$$

Define

$$\begin{aligned} D(\varphi(A)) &= \{u \in H : \int |\varphi(\lambda)|^2 \, d\mu_i(\lambda) < \infty\} \\ &= \left\{ u \in H : \sup_{|\psi| \le |\varphi|, \psi \in \mathrm{Ba}_b} \|\psi(A)u\| < \infty \right\}. \end{aligned}$$

This is a linear subspace of H.

To define $\varphi(A)$, let $\varphi_j \in \mathrm{Ba}_b$ be such that $\varphi_j \to \varphi$ with $|\varphi_j| \le |\varphi|$. Then, for any $u \in D(\varphi(A))$, $\lim_{j\to\infty} \varphi_j(A)u$ exists ($\|\varphi_j(A)u - \varphi_k(A)u\|^2 = \int |\varphi_j - \varphi_k|^2 \, d\mu_i \xrightarrow{j,k\to\infty} 0$ by dominated convergence). So we let

$$\varphi(A)u = \lim \varphi_k(A)u,$$

which is independent of the chosen sequence.

Proposition 1.2. *Let A be self-adjoint, and let $\varphi \in \mathrm{Ba}(\mathbb{R})$. Then $\varphi(A)$ is densely defined, and $\varphi(A)^* = \overline{\varphi}(A)$ (so $D(\varphi(A)^*) = D(\overline{\varphi})$). In particular, $\varphi(A)$ is closed.*